Elektromagnetische Feldtheorie I (EFT I) / Electromagnetic Field Theory I (EFT I)

Lecture 9 / 9. Vorlesung

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Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono-, Di- und Quadrupol ... (2)

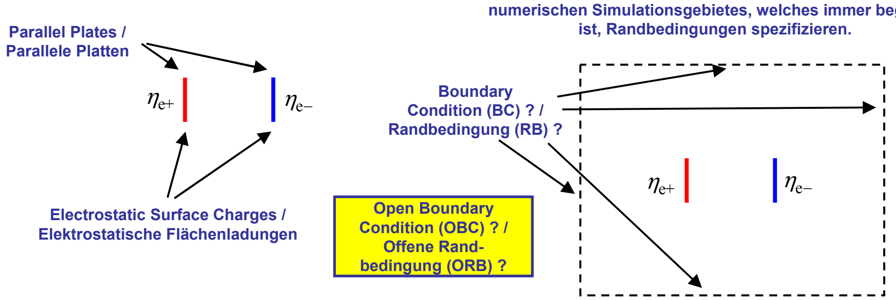
**Application: Numerical Solution of Unbounded Static Problems /
Anwendung: Numerische Lösung von unbegrenzten statischen Problemen**

$$\Delta\Phi_e(\underline{\mathbf{R}}) = -\frac{\rho_e(\underline{\mathbf{R}})}{\varepsilon_0}$$

**Problem: Parallel Plate Capacitor in an Unbounded Region /
Problem: Paralleler Plattenkondensator in einem unbegrenzten Gebiet**

**Outline of the Problem /
Entwurf des Problems**

**Numerical Solution: We need to Specify Boundary Conditions at the Boundaries of the Simulation Area which is always bounded. /
Numerische Lösung: Wir müssen für die Ränder des numerischen Simulationsgebietes, welches immer begrenzt ist, Randbedingungen spezifizieren.**



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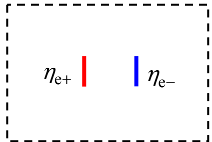
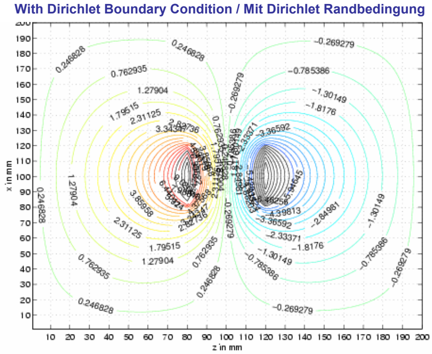
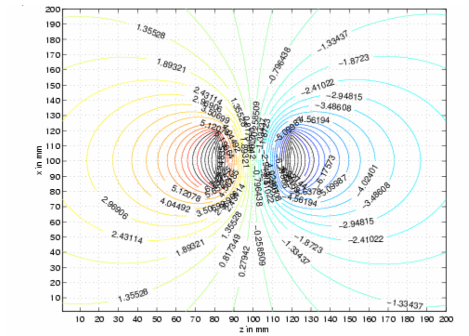


With Dirichlet Boundary Condition / Mit Dirichlet Randbedingung



With Open Boundary Condition (OBC) / Mit offener Randbedingung (ORB)

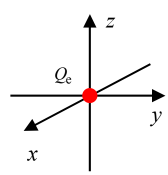


Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Point Charge(s): Monopole, Dipole, and Quadrupole / Punktladung(en): Mono-, Di- und Quadrupol

Monopole Moment / Monopolmoment

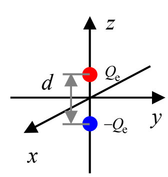
One Point Charge / Eine Punktladung



$$Q_e \neq 0, \quad \underline{\mathbf{p}}_e = \underline{\mathbf{0}}, \quad \underline{\underline{\mathbf{q}}}_e = \underline{\underline{\mathbf{0}}}$$

Dipole Moment / Dipolmoment

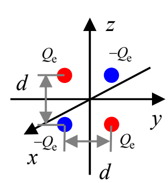
Two Point Charges / Zwei Punktladungen



$$Q_e = 0, \quad \underline{\mathbf{p}}_e \neq \underline{\mathbf{0}}, \quad \underline{\underline{\mathbf{q}}}_e = \underline{\underline{\mathbf{0}}}$$

Quadrupole Moment/ Quadrupolmoment

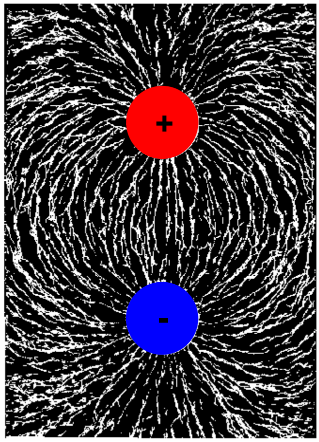
Four Point Charges / Vier Punktladungen



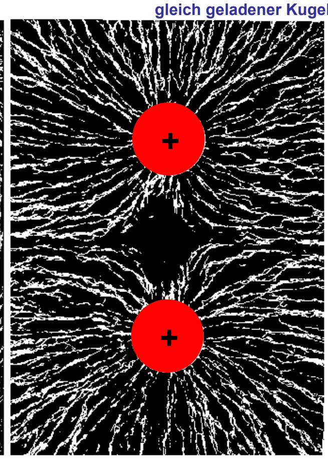
$$Q_e = 0, \quad \underline{\mathbf{p}}_e = \underline{\mathbf{0}}, \quad \underline{\underline{\mathbf{q}}}_e \neq \underline{\underline{\mathbf{0}}}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Field Lines of the Electric Field Strength of Two Spheres Carrying Charges of Opposite Sign / Feldlinien der elektrischen Feldstärke zweier ungleich geladener Kugeln



Electric Field Lines of Two Spheres Carrying Charges of the Same Sign / Feldlinien der elektrischen Feldstärke zweier gleich geladener Kugeln



The Field Lines - Lines of Force – of the Electrostatic Fields were Formed by Grass Seeds (Kentucky Blue Grass) Strewed on Glass Plates. To Make the Seeds more Mobile, the Plates were Waxed with a Liquid Wax. /
Die Feldlinien – Kraftlinien – elektrostatischer Felder formiert durch Grassamen (Kentucky Blue Grass) gestreut auf Glasplatten. Um den Samen mobiler zu machen, wurden die Glasplatten mit flüssigem Wachs behandelt.

Reference: Oleg D. Jefimenko: Electricity and Magnetism. An Introduction to the Theory of Electric and Magnetic Fields. Electret Scientific Company, 1989 (reprint of the 1966 edition)

Electrostatic Field Lines / Elektrostatische Feldlinien

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Kentucky Blue Grass (Bot.), a Valuable Pasture and Meadow Grass (Poa Pratensis), found in Both Europe and America. A Species of Grass which has Running Rootstocks and Spreads Rapidly. It is Valuable as a Pasture Grass, as it Endures both Winter and Drought Better than Other Kinds, and is very Nutritious. /
Kentucky Blue Grass (Bot.), ist ein wertvolles Weide- und Wiesengras (Poa Pratensis), welches man in Europa und Amerika vorfindet. Es ist eine Grasart, welches einen laufenden Wurzelstock hat und sich schnell ausbreitet. Es ist sehr wertvoll als Weidegras, weil es Winter als auch Dürre besser als andere Arten übersteht, und es ist sehr nahrhaft.

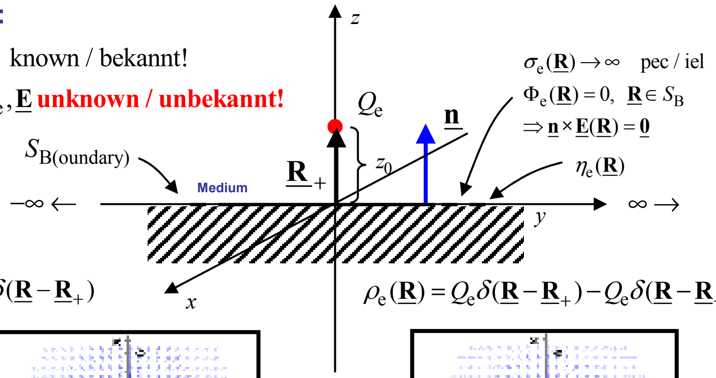
Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode

Problem:

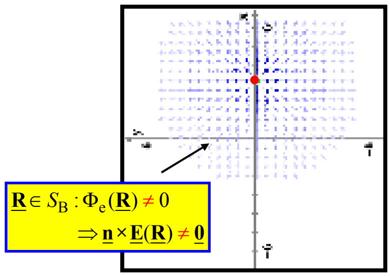
Q_e known / bekannt!

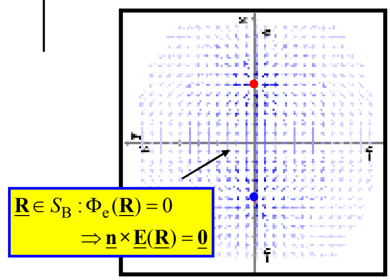
$\Phi_e, \underline{\mathbf{E}}$ **unknown / unbekannt!**



$\rho_e(\underline{\mathbf{R}}) = Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+)$

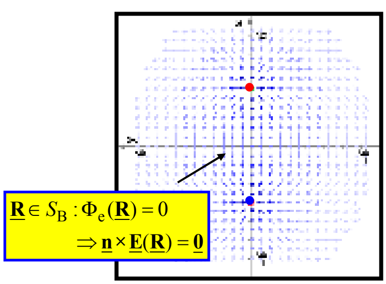
$\rho_e(\underline{\mathbf{R}}) = Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+) - Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-)$



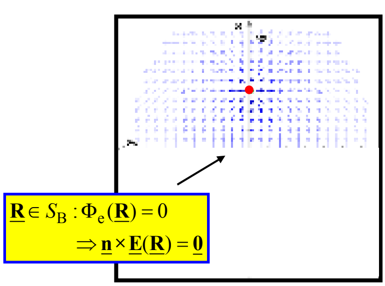


Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode



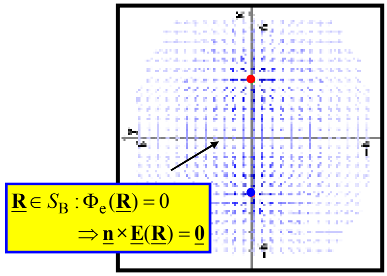
$$\Phi_e(\underline{\mathbf{R}}) = \frac{Q_e}{4\pi\varepsilon_0}\left(\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|}\right)$$



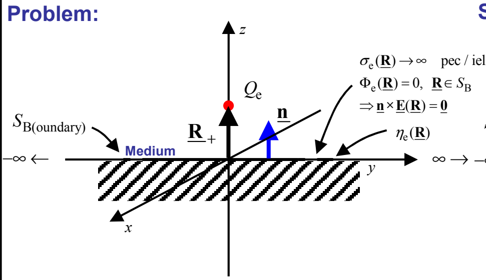
$$\Phi_e(\underline{\mathbf{R}}) = \begin{cases} \dfrac{Q_e}{4\pi\varepsilon_0}\left(\dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|}\right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

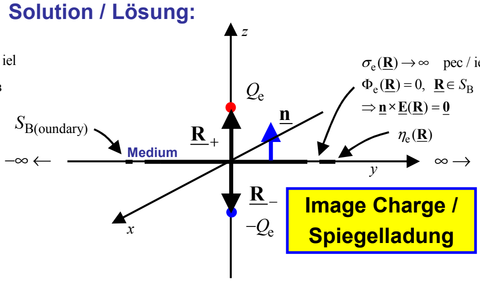
Method of Images / Spiegelungsmethode



Problem:



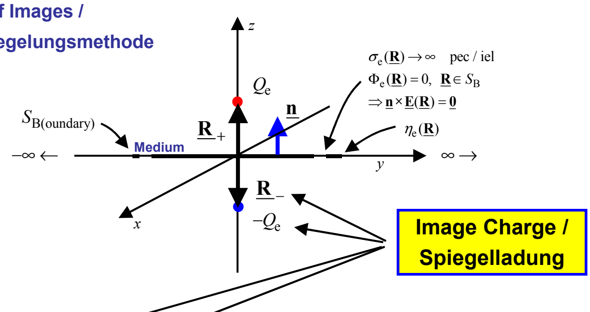
Solution / Lösung:



Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode

Solution by Applying the Method of Images /
Lösung durch Anwendung der Spiegelungsmethode



$$\rho_e(\underline{\mathbf{R}}) = Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+) - Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-)$$

with / mit $\quad \underline{\mathbf{R}}_+ = z_0 \underline{\mathbf{e}}_z \qquad \underline{\mathbf{R}}_- = -\underline{\mathbf{R}}_+ = -z_0 \underline{\mathbf{e}}_z$

$$\Phi_e(\underline{\mathbf{R}}) = \begin{cases} \dfrac{Q_e}{4\pi\varepsilon_0}\left(\dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|}\right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode

$$\rho_\mathrm{e}(\underline{\mathbf{R}}) = Q_\mathrm{e}\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_+) - Q_\mathrm{e}\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_-)$$

with / mit

$$\underline{\mathbf{R}}_+ = z_0\underline{\mathbf{e}}_z \qquad \underline{\mathbf{R}}_- = -\underline{\mathbf{R}}_+ = -z_0\underline{\mathbf{e}}_z$$

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi\varepsilon_0}\left(\dfrac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|} - \dfrac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|}\right) & z \ge 0 \\ 0 & z < 0 \end{cases}$$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})$$

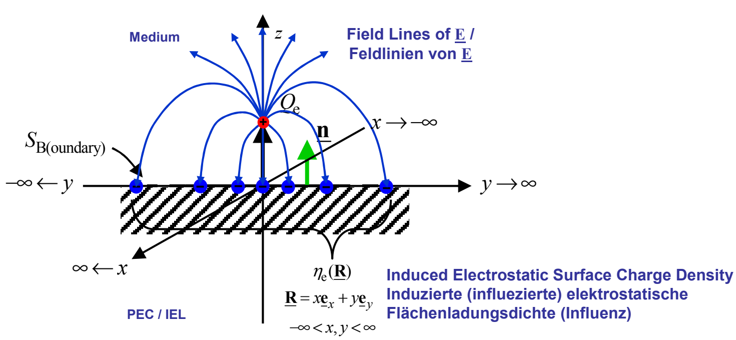
$$= \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi\varepsilon_0}\left(\dfrac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_+}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \dfrac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_-}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right) & z \ge 0 \\ 0 & z < 0 \end{cases}$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \varepsilon_0\underline{\mathbf{E}}(\underline{\mathbf{R}})$$

$$= \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi}\left(\dfrac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_+}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \dfrac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_-}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right) & z \ge 0 \\ 0 & z < 0 \end{cases}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode



Without the Method of Images we have to Solve the Following Integral Equation for the Unknown Induced Electrostatic Surface Charge / Ohne die Spiegelungsmethode muss man die folgende Integralgleichung für die induzierte (influezierte) elektrostatische Flächenladungsdichte lösen

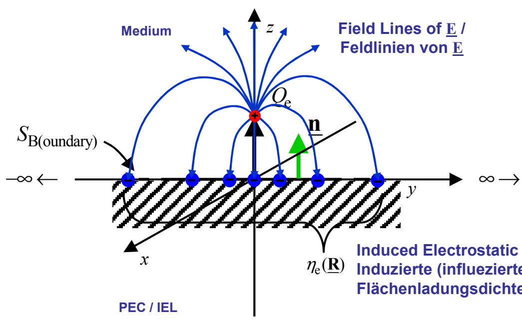
Unknown / Unbekannt

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_\mathrm{e}}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|} + \iint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \frac{\eta_\mathrm{e}(\underline{\mathbf{R}}')}{|\underline{\mathbf{R}}'-\underline{\mathbf{R}}_+|}\, d^2\underline{\mathbf{R}}'\Bigg|_{z=0}\right] = 0$$

for / für $\Phi_\mathrm{e}(\underline{\mathbf{R}})\big|_{z=0} = 0$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode



If D is known from the Method of Images / Falls D über die Spiegelungsmethode bekannt ist

$$\underline{\mathbf{D}}(\underline{\mathbf{R}})\Big|_{\underline{\mathbf{R}}\in S_B} = \begin{matrix}\text{known}\\\text{bekannt}\end{matrix} \;!$$

$\eta_e(\underline{\mathbf{R}})$ is Defined by the Normal Component of D / $\eta_e(\underline{\mathbf{R}})$ ist definiert über die Normalkomponente von D

$$\eta_e(\underline{\mathbf{R}}) = \underline{\mathbf{n}}\bullet\underline{\mathbf{D}}(\underline{\mathbf{R}})\Big|_{\underline{\mathbf{R}}\in S_B}$$

$$= \underline{\mathbf{n}}\bullet\frac{Q_e}{4\pi}\left[\frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_+}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_-}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right]\Bigg|_{\underline{\mathbf{R}}\in S_B} \quad \begin{matrix}\text{for}\\\text{für}\end{matrix} \quad z=0$$

$$= \frac{Q_e}{4\pi}\underline{\mathbf{e}}_z\bullet\left[\frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_+}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_-}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right]\Bigg|_{z=0}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode

$$\eta_e(\underline{\mathbf{R}}) = \underline{\mathbf{n}}\bullet\underline{\mathbf{D}}(\underline{\mathbf{R}})\Big|_{\underline{\mathbf{R}}\in S_B}$$

$$= \frac{Q_e}{4\pi}\left[\frac{\underline{\mathbf{e}}_z\bullet(\underline{\mathbf{R}}-\underline{\mathbf{R}}_+)}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \frac{\underline{\mathbf{e}}_z\bullet(\underline{\mathbf{R}}-\underline{\mathbf{R}}_-)}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right]\Bigg|_{z=0}$$

$$= \frac{Q_e}{4\pi}\left[\frac{z-z_0}{\left[x^2+y^2+(z-z_0)^2\right]^{3/2}} - \frac{z+z_0}{\left[x^2+y^2+(z-z_0)^2\right]^{3/2}}\right]\Bigg|_{z=0}$$

$$= \frac{Q_e}{4\pi}\left[\frac{-z_0}{\left[x^2+y^2+z_0^2\right]^{3/2}} - \frac{z_0}{\left[x^2+y^2+z_0^2\right]^{3/2}}\right]$$

$$= -\frac{Q_e}{2\pi}\frac{z_0}{\left[x^2+y^2+z_0^2\right]^{3/2}}$$

$$= -\frac{Q_e}{2\pi}\frac{z_0}{\left[r^2+z_0^2\right]^{3/2}}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode

$$\eta_e(\underline{\mathbf{R}}) = \underline{\mathbf{n}} \bullet \underline{\mathbf{D}}(\underline{\mathbf{R}})\Big|_{\underline{\mathbf{R}} \in S_B} = -\frac{Q_e}{2\pi} \frac{z_0}{\left[r^2 + z_0^2\right]^{3/2}}$$

Total Electric Charge at the xy Plane at z=0 / Gesamtladung auf der xy Ebene bei z=0

$$Q_e^{tot} = -\frac{Q_e}{2\pi} \int_{\varphi=0}^{2\pi} \int_{r=0}^{\infty} \frac{z_0}{\left[r^2 + z_0^2\right]^{3/2}} r\,dr\,d\varphi$$

$$= -\frac{Q_e}{2\pi} \int_{r=0}^{\infty} \frac{z_0}{\left[r^2 + z_0^2\right]^{3/2}} r\,dr \underbrace{\int_{\varphi=0}^{2\pi} d\varphi}_{=2\pi}$$

$$= -Q_e z_0 \int_{r=0}^{\infty} \frac{r}{\left[r^2 + z_0^2\right]^{3/2}} dr$$

$$\int \frac{x}{\left[x^2 + a^2\right]^{3/2}} dx = -\frac{1}{\sqrt{x^2 + a^2}}$$

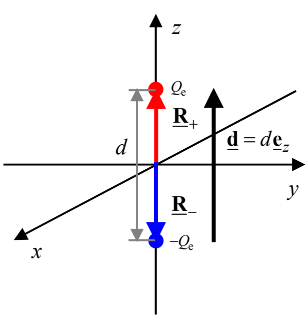
$$= Q_e z_0 \left[\underbrace{\frac{1}{\sqrt{(r \to \infty)^2 + z_0^2}}}_{\to 0} - \frac{1}{z_0} \right]$$

$$= -Q_e$$

$$Q_e^{tot} = -Q_e$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Electrostatic Dipole / Elektrostatischer Dipol



Electrostatic Volume Charge Density / Elektrostatische Raumladungsdichte

$$\rho_e(\underline{\mathbf{R}}) = Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+) - Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-) = Q_e \delta(\underline{\mathbf{R}} - \frac{d}{2}\underline{\mathbf{e}}_z) - Q_e \delta(\underline{\mathbf{R}} + \frac{d}{2}\underline{\mathbf{e}}_z)$$

with $\underline{\mathbf{R}}_+ = \frac{d}{2}\underline{\mathbf{e}}_z$, $\underline{\mathbf{R}}_- = -\frac{d}{2}\underline{\mathbf{e}}_z$

Electrostatic Potential / Elektrostatisches Potential

$$\Phi_e(\underline{\mathbf{R}}) = \frac{Q_e}{4\pi\varepsilon_0} \left(\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|} \right)$$

Electrostatic Field Strength / Elektrostatische Feldstärke

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = \frac{Q_e}{4\pi\varepsilon_0} \left(\frac{\underline{\mathbf{R}} - \underline{\mathbf{R}}_+}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|^3} - \frac{\underline{\mathbf{R}} - \underline{\mathbf{R}}_-}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|^3} \right)$$

Electrostatic Dipole Moment / Elektrische Dipolmoment

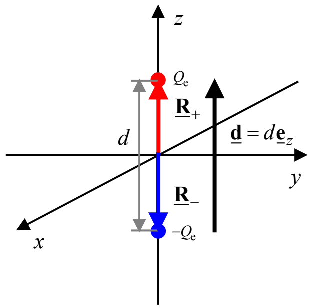
$$\underline{\mathbf{p}}_e = \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}')\underline{\mathbf{R}}'\,d^3\underline{\mathbf{R}}' = \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left[Q_e \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_+) - Q_e \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_-) \right] \underline{\mathbf{R}}'\,d^3\underline{\mathbf{R}}'$$

$$= Q_e \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_+)\underline{\mathbf{R}}'\,d^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}_+} - Q_e \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_-)\underline{\mathbf{R}}'\,d^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}_-} = Q_e \underline{\mathbf{R}}_+ - Q_e \underline{\mathbf{R}}_- = Q_e(\underline{\mathbf{R}}_+ - \underline{\mathbf{R}}_-) = Q_e \underline{\mathbf{d}}$$

Distance Vector / Abstandsvektor $\underline{\mathbf{d}} = \underline{\mathbf{R}}_+ - \underline{\mathbf{R}}_- = \frac{d}{2}\underline{\mathbf{e}}_z + \frac{d}{2}\underline{\mathbf{e}}_z = d\underline{\mathbf{e}}_z$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Electrostatic Dipole / Elektrostatischer Dipol



Electrostatic Dipole Moment / Elektrostatisches Dipolmoment

$$\underline{\mathbf{p}}_e = Q_e \underline{\mathbf{d}} = p_e \hat{\underline{\mathbf{p}}}_e$$

with / mit

$$p_e = Q_e |\underline{\mathbf{d}}| = Q_e |\underline{\mathbf{R}}_+ + \underline{\mathbf{R}}_-|$$

$$\hat{\underline{\mathbf{p}}}_e = \hat{\underline{\mathbf{d}}} = \widehat{|\underline{\mathbf{R}}_+ + \underline{\mathbf{R}}_-|}$$

Electrostatic Quadrupole Moment / Elektrostatisches Quadrupolmoment

$$\begin{aligned}
\underline{\underline{\mathbf{q}}}_e &= \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}')\underline{\mathbf{R}}'\underline{\mathbf{R}}' d^3\underline{\mathbf{R}}' \\
&= \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left[Q_e\delta(\underline{\mathbf{R}}'-\underline{\mathbf{R}}_+) - Q_e\delta(\underline{\mathbf{R}}'-\underline{\mathbf{R}}_-)\right]\underline{\mathbf{R}}'\underline{\mathbf{R}}' d^3\underline{\mathbf{R}}' \\
&= Q_e \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \delta(\underline{\mathbf{R}}'-\underline{\mathbf{R}}_+)\underline{\mathbf{R}}'\underline{\mathbf{R}}' d^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}_+\underline{\mathbf{R}}_+} - Q_e \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \delta(\underline{\mathbf{R}}'-\underline{\mathbf{R}}_-)\underline{\mathbf{R}}'\underline{\mathbf{R}}' d^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}_-\underline{\mathbf{R}}_-} \\
&= Q_e\underline{\mathbf{R}}_+\underline{\mathbf{R}}_+ - Q_e\underline{\mathbf{R}}_-\underline{\mathbf{R}}_- \\
&= Q_e\left[\left(\frac{d}{2}\underline{\mathbf{e}}_z\right)\left(\frac{d}{2}\underline{\mathbf{e}}_z\right) - \left(-\frac{d}{2}\underline{\mathbf{e}}_z\right)\left(-\frac{d}{2}\underline{\mathbf{e}}_z\right)\right] \\
&= Q_e\underbrace{\left[d\underline{\mathbf{e}}_z\underline{\mathbf{e}}_z - d\underline{\mathbf{e}}_z\underline{\mathbf{e}}_z\right]}_{=\underline{\underline{\mathbf{0}}}} \\
&= \underline{\underline{\mathbf{0}}}
\end{aligned}$$

Magnetostatic (MS) Fields / Magnetostatische (MS) Felder

Governing Equations / Grundgleichungen

Integral Form / Integralform

$$\oint_{C=\partial S} \underline{\mathbf{H}}(\underline{\mathbf{R}})\bullet d\underline{\mathbf{R}} = \iint_S \underline{\mathbf{J}}_e(\underline{\mathbf{R}})\bullet d\underline{\mathbf{S}}$$

$$\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}})\bullet d\underline{\mathbf{S}} = 0$$

Differential Form / Differentialform

$$\nabla\times\underline{\mathbf{H}}(\underline{\mathbf{R}}) = \underline{\mathbf{J}}_e(\underline{\mathbf{R}})$$

$$\nabla\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}}) = 0$$

$\underline{\mathbf{J}}_e(\underline{\mathbf{R}})$

is a Known Prescribed Electric Current Density: For Example a Electric Current Density in a Wire / ist eine bekannte vorgegebene elektrische Stromdichte: Zum Beispiel eine elektrische Stromdichte in einem Draht

Because of / Weil $\nabla\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}}) = 0$ $\underline{\mathbf{B}}$ Can be Represented by / $\underline{\mathbf{B}}$ kann dargestellt werden über $\underline{\mathbf{B}}(\underline{\mathbf{R}}) = \nabla\times\underline{\mathbf{A}}(\underline{\mathbf{R}})$

In Vacuum we have / Im Vakuum gilt

$$\underline{\mathbf{B}}(\underline{\mathbf{R}}) = \mu_0\underline{\mathbf{H}}(\underline{\mathbf{R}})$$

$$\underline{\mathbf{H}}(\underline{\mathbf{R}}) = \frac{1}{\mu_0}\underline{\mathbf{B}}(\underline{\mathbf{R}}) = \frac{1}{\mu_0}\nabla\times\underline{\mathbf{A}}(\underline{\mathbf{R}})$$

Magnetostatic (MS) Fields / Magnetostatische (MS) Felder

$$\nabla \bullet \underline{\mathbf{A}}(\underline{\mathbf{R}}) = 0$$

Coulomb Gauge / Coulomb-Eichung

It follows / Es folgt

$$\mu_0 \underline{\mathbf{H}}(\underline{\mathbf{R}}) = \nabla \times \underline{\mathbf{A}}(\underline{\mathbf{R}})$$

Applying the Curl Operator Gives / Die Anwendung des Rotationsoperators ergibt

$$\mu_0 \nabla \times \underline{\mathbf{H}}(\underline{\mathbf{R}}) = \nabla \times \nabla \times \underline{\mathbf{A}}(\underline{\mathbf{R}}) = \mu_0 \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}})$$

$$\nabla \times \nabla \times \underline{\mathbf{A}}(\underline{\mathbf{R}}) = \nabla \underbrace{\nabla \bullet \underline{\mathbf{A}}(\underline{\mathbf{R}})}_{=0} - \nabla \bullet \nabla \underline{\mathbf{A}}(\underline{\mathbf{R}}) = -\nabla \bullet \nabla \underline{\mathbf{A}}(\underline{\mathbf{R}}) = -\Delta \underline{\mathbf{A}}(\underline{\mathbf{R}})$$

$$\boxed{\Delta \underline{\mathbf{A}}(\underline{\mathbf{R}}) = -\mu_0 \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}})}$$

Vector Possion's Equation / Vektorielle Possion-Gleichung

Magnetostatic (MS) Fields / Magnetostatische (MS) Felder

$$\Delta \underline{\mathbf{A}}(\underline{\mathbf{R}}) = -\mu_0 \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}})$$

Vector Possion's Equation / Vektorielle Possion-Gleichung

Solution / Lösung

$$\underline{\mathbf{A}}(\underline{\mathbf{R}}) = \mu_0 \iiint_{V_S} G(\underline{\mathbf{R}} - \underline{\mathbf{R}}') \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}}')\, d^3\underline{\mathbf{R}}'$$

With the Three-Dimensional Static Green's Function / Mit der dreidimensionalen statischen Greenschen Funktion

$$G(\underline{\mathbf{R}} - \underline{\mathbf{R}}') = \frac{1}{4\pi} \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|}$$

$$\underline{\mathbf{H}}(\underline{\mathbf{R}}) = \frac{1}{\mu_0} \nabla \times \underline{\mathbf{A}}(\underline{\mathbf{R}}) = \frac{1}{\mu_0} \nabla \times \left[\mu_0 \iiint_{V_S} G(\underline{\mathbf{R}} - \underline{\mathbf{R}}') \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}}')\, d^3\underline{\mathbf{R}}' \right] = \iiint_{V_S} \nabla \times \left[G(\underline{\mathbf{R}} - \underline{\mathbf{R}}') \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}}') \right] d^3\mathbf{R}'$$

$$\nabla \times (\Phi \underline{\mathbf{A}}) = \Phi \nabla \times \underline{\mathbf{A}} + \nabla \Phi \times \underline{\mathbf{A}}$$

$$\nabla \times [G(\underline{\mathbf{R}} - \underline{\mathbf{R}}') \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}}')] = G(\underline{\mathbf{R}} - \underline{\mathbf{R}}') \underbrace{\nabla \times \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}}')}_{=\underline{\mathbf{0}}} + \nabla G(\underline{\mathbf{R}} - \underline{\mathbf{R}}') \times \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}}')$$

Magnetostatic (MS) Fields / Magnetostatische (MS) Felder

$$\nabla G(\underline{\mathbf{R}}-\underline{\mathbf{R}}') = \nabla \frac{1}{4\pi}\frac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|} = \frac{1}{4\pi}\nabla\frac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|} = -\frac{1}{4\pi}\frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}'}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|^3}$$

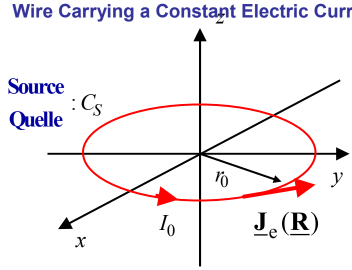
$$\begin{aligned}\underline{\mathbf{H}}(\underline{\mathbf{R}}) &= -\iiint_{V_S}\frac{1}{4\pi}\frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}'}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|^3}\times\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}}')d^3\underline{\mathbf{R}}' \\ &= \iiint_{V_S}\frac{1}{4\pi}\frac{\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}}')\times(\underline{\mathbf{R}}-\underline{\mathbf{R}}')}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|^3}d^3\underline{\mathbf{R}}' \\ &= \frac{1}{4\pi}\iiint_{V_S}\frac{\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}}')\times(\underline{\mathbf{R}}-\underline{\mathbf{R}}')}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|^3}d^3\underline{\mathbf{R}}'\end{aligned}$$

$$\boxed{\underline{\mathbf{H}}(\underline{\mathbf{R}}) = \frac{1}{4\pi}\iiint_{V_S}\frac{\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}}')\times(\underline{\mathbf{R}}-\underline{\mathbf{R}}')}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|^3}d^3\underline{\mathbf{R}}'}$$

Biot-Savart's Law (for a Given Volume Source) / Biot-Savartsches Gesetz (für eine Volumenquelle)

Magnetostatic (MS) Fields / Magnetostatische (MS) Felder

Biot-Savart's Law for a Line Source / Biot-Savartsches Gesetz für eine Linienquelle
Wire Carrying a Constant Electric Current I0 / Biot-Savartsches Gesetz für eine Linienquelle



$$\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}}) = I_0\,\delta(r-r_0)\,\delta(z)\,\underline{\mathbf{e}}_\varphi, \qquad r_0>0$$

$$\begin{aligned}\underline{\mathbf{H}}(\underline{\mathbf{R}}) &= \frac{1}{4\pi}\iiint_{V_S}\frac{\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}}')\times(\underline{\mathbf{R}}-\underline{\mathbf{R}}')}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|^3}d^3\underline{\mathbf{R}}' \\ &= \frac{1}{4\pi}\int_{z'=-\infty}^{\infty}\int_{\varphi'=0}^{2\pi}\int_{r'=0}^{\infty}\frac{I_0\delta(r'-r_0)\delta(z')\underline{\mathbf{e}}_{\varphi'}(\varphi')\times(\underline{\mathbf{R}}-\underline{\mathbf{R}}')}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|^3}r'dr'd\varphi'dz' \\ &= \frac{I_0}{4\pi}\int_{\varphi'=0}^{2\pi}\int_{r'=0}^{\infty}\frac{\delta(r'-r_0)\underline{\mathbf{e}}_{\varphi'}(\varphi')\times(\underline{\mathbf{R}}-\underline{\mathbf{R}}')}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|^3}r'dr'd\varphi'\Bigg|_{\underline{\mathbf{R}}':z'=0}\end{aligned}$$

Magnetostatic (MS) Fields / Magnetostatische (MS) Felder

Biot-Savart's Law for a Line Source / Biot-Savartsches Gesetz für eine Linienquelle

$$\underline{\mathbf{H}}(\underline{\mathbf{R}}) = \frac{I_0}{4\pi}\int_{\varphi'=0}^{2\pi} \frac{\overbrace{\underline{\mathbf{e}}_{\varphi'}(\varphi')\, r_0\, d\varphi'}^{=\mathbf{d}\underline{\mathbf{R}}'} \times (\underline{\mathbf{R}}-\underline{\mathbf{R}}')}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|^3}\Bigg|_{\underline{\mathbf{R}}':r'=r_0;z'=0}$$

$$= \frac{I_0}{4\pi}\int_{\varphi'=0}^{2\pi} \frac{\mathbf{d}\underline{\mathbf{R}}' \times (\underline{\mathbf{R}}-\underline{\mathbf{R}}')}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|^3}\Bigg|_{\underline{\mathbf{R}}':r'=r_0;z'=0}$$

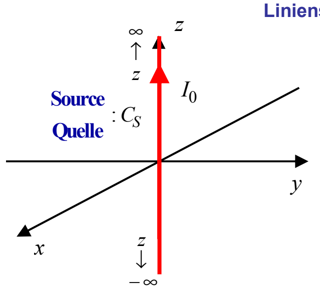
$$= \frac{I_0}{4\pi}\int_{C_S} \frac{\mathbf{d}\underline{\mathbf{R}}' \times (\underline{\mathbf{R}}-\underline{\mathbf{R}}')}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|^3}\Bigg|_{\underline{\mathbf{R}}':r'=r_0;z'=0}$$

$$\boxed{\underline{\mathbf{H}}(\underline{\mathbf{R}}) = \frac{I_0}{4\pi}\int_{C_S} \frac{\mathbf{d}\underline{\mathbf{R}}' \times (\underline{\mathbf{R}}-\underline{\mathbf{R}}')}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|^3}}$$

Biot-Savart Law for a Line Source with Contour C_S / Biot-Savartsches Gesetz für eine Linienquelle mit der Kontur C_S

Magnetostatic (MS) Fields / Magnetostatische (MS) Felder

Example: Magnetostatic Field of a Infinite Thin and Long Wire Carrying an Electric Line Current / Beispiel: Magnetostatische Feld eines unendlich dünnen und langen Drahtes, der einen elektrischen Linienstrom führt (1)



$$\underline{\mathbf{H}}(\underline{\mathbf{R}}) = \frac{I_0}{4\pi}\int_{C_S} \frac{\mathbf{d}\underline{\mathbf{R}}' \times (\underline{\mathbf{R}}-\underline{\mathbf{R}}')}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|^3}$$

$$\underline{\mathbf{R}} = r\underline{\mathbf{e}}_r(\varphi) + z\underline{\mathbf{e}}_z$$

$$\underline{\mathbf{R}}' = r'\underline{\mathbf{e}}_{r'}(\varphi') + z'\underline{\mathbf{e}}_z\Big|_{r'=0}$$

$$= z'\underline{\mathbf{e}}_z, \qquad -\infty \le z' \le \infty$$

$$\underline{\mathbf{R}} - \underline{\mathbf{R}}' = r\underline{\mathbf{e}}_r + (z-z')\underline{\mathbf{e}}_z$$

$$\mathbf{d}\underline{\mathbf{R}}' = \frac{d}{dz'}\underline{\mathbf{R}}' dz'$$

$$= \underline{\mathbf{e}}_z dz'$$

$$\mathbf{d}\underline{\mathbf{R}}' \times (\underline{\mathbf{R}}-\underline{\mathbf{R}}') = \begin{vmatrix} \underline{\mathbf{e}}_r & \underline{\mathbf{e}}_\varphi(\varphi) & \underline{\mathbf{e}}_z \\ 0 & 0 & dz' \\ r & 0 & z-z' \end{vmatrix}$$

$$= r\,\underline{\mathbf{e}}_\varphi\, dz'$$

$$\left|\underline{\mathbf{R}} - \underline{\mathbf{R}}'\right|^3 = \left[r^2 + (z-z')^2\right]$$

Magnetostatic (MS) Fields / Magnetostatische (MS) Felder

Example: Magnetostatic Field of a Infinite Thin and Long Wire Carrying an Electric Line Current / Beispiel: Magnetostatische Feld eines unendlich dünnen und langen Drahtes, der einen elektrischen Linienstrom führt (2)

$$\underline{\mathbf{H}}(\underline{\mathbf{R}}) = \frac{I_0}{4\pi} \int_{z'=-\infty}^{\infty} \frac{r\underline{\mathbf{e}}_\varphi(\varphi)}{\left[r^2 + (z-z')^2\right]^{3/2}} dz'$$

$$= -\frac{I_0}{4\pi} r\underline{\mathbf{e}}_\varphi(\varphi) \int_{\alpha=\infty}^{-\infty} \frac{d\alpha}{\left[r^2+\alpha^2\right]^{3/2}}$$

$$= \frac{I_0}{4\pi} r\underline{\mathbf{e}}_\varphi(\varphi) \int_{\alpha=-\infty}^{\infty} \frac{d\alpha}{\left[r^2+\alpha^2\right]^{3/2}}$$

$$= \frac{I_0}{4\pi} r\underline{\mathbf{e}}_\varphi(\varphi) \int_{\alpha=-\infty}^{\infty} \frac{d\alpha}{\left[r^2+\alpha^2\right]^{3/2}}$$

$$= \frac{I_0}{4\pi} r\underline{\mathbf{e}}_\varphi(\varphi) \left. \frac{\alpha}{r^2\sqrt{r^2+\alpha^2}} \right|_{\alpha=-\infty}^{\infty}$$

With the Substitution / Mit der Substitution

$$\alpha = z - z'$$

$$dz' = -d\alpha$$

$$z' = -\infty: \quad \alpha = z + \infty = \infty$$

$$z' = \infty: \quad \alpha = z - \infty = -\infty$$

$$\int \frac{dx}{\left[ax^2+b\right]^{3/2}} = \frac{x}{b\sqrt{ax^2+b}}$$

with
mit $a = 1,\ b = r^2$

Magnetostatic (MS) Fields / Magnetostatische (MS) Felder

Example: Magnetostatic Field of a Infinite Thin and Long Wire Carrying an Electric Line Current / Beispiel: Magnetostatische Feld eines unendlich dünnen und langen Drahtes, der einen elektrischen Linienstrom führt (3)

$$\underline{\mathbf{H}}(\underline{\mathbf{R}}) = \frac{I_0}{4\pi}\frac{1}{r} \left. \frac{\operatorname{sgn}(\alpha)\,|\alpha|}{\sqrt{r^2+\alpha^2}} \right|_{\alpha=-\infty}^{\infty} \underline{\mathbf{e}}_\varphi(\varphi)$$

$$= \frac{I_0}{4\pi}\frac{1}{r} \left. \frac{\operatorname{sgn}(\alpha)\,|\alpha|}{|\alpha|\sqrt{1+r^2/\alpha^2}} \right|_{\alpha=-\infty}^{\infty} \underline{\mathbf{e}}_\varphi(\varphi)$$

$$= \frac{I_0}{4\pi}\frac{1}{r} \left[\underbrace{\lim_{\alpha\to\infty} \frac{\operatorname{sgn}(\alpha)}{\sqrt{1+r^2/\alpha^2}}}_{=1} - \underbrace{\lim_{\alpha\to-\infty} \frac{\operatorname{sgn}(\alpha)}{\sqrt{1+r^2/\alpha^2}}}_{=-1} \right] \underline{\mathbf{e}}_\varphi(\varphi)$$

$$= \frac{I_0}{4\pi}\frac{1}{r} \underbrace{[1-(-1)]}_{2} \underline{\mathbf{e}}_\varphi(\varphi)$$

$$= \frac{I_0}{2\pi}\frac{1}{r} \underline{\mathbf{e}}_\varphi(\varphi)$$

With the Signum Function / Mit der Signum-Funktion

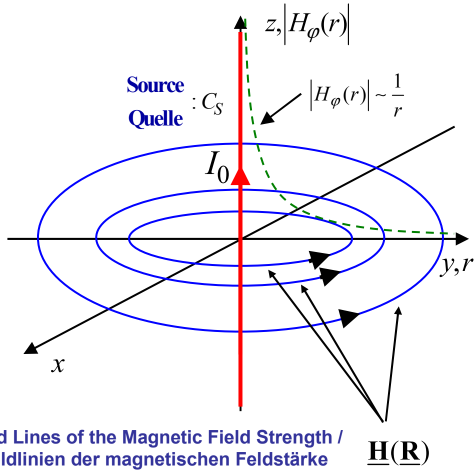
$$\operatorname{sgn}(\alpha) = \begin{cases} -1 & \alpha < 0 \\ 1 & \alpha > 0 \end{cases}$$

$$\alpha = \operatorname{sgn}(\alpha)\ |\alpha|$$

With / Mit $\sqrt{\alpha^2} = |\alpha|$

Magnetostatic (MS) Fields / Magnetostatische (MS) Felder

Example: Magnetostatic Field of a Infinite Thin and Long Wire Carrying an Electric Line Current / Beispiel: Magnetostatische Feld eines unendlich dünnen und langen Drahtes, der einen elektrischen Linienstrom führt (3)



$$\underline{\mathbf{H}}(\underline{\mathbf{R}}) = \underbrace{\frac{I_0}{2\pi}\frac{1}{r}}_{=H_\varphi(r)} \underline{\mathbf{e}}_\varphi(\varphi)$$

$$= H_\varphi(r)\underline{\mathbf{e}}_\varphi(\varphi)$$

$$|\underline{\mathbf{H}}(\underline{\mathbf{R}})| = |H_\varphi(r)| \sim \frac{1}{r}$$

End of Lecture 9 /
Ende der 9. Vorlesung